Studying some Markov chains using representation theory of monoids

Nicolas M. Thiéry

Laboratoire de Recherche en Informatique
Université Paris Sud 11

ALÉA 2014, March 18th of 2014

Joint work with:
Arvind Ayyer, Benjamin Steinberg, Anne Schilling
arXiv: 1305.1697, 1401.4250

Highlight of the talk

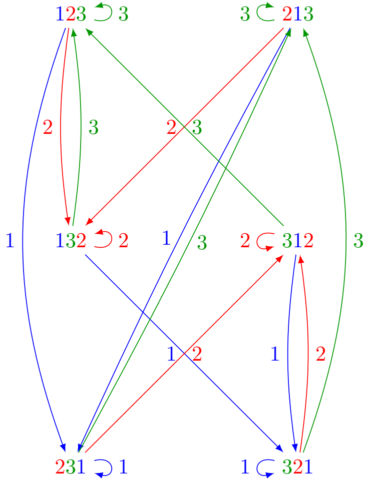
A first example: the Tsetlin library

Configuration: n books on a shelf
Operation T_i: move the i-th book to the right

A first example: the Tsetlin library

Configuration: n books on a shelf

Operation T_i: move the i-th book to the right



A first example: the Tsetlin library

Configuration: n books on a shelf
Operation T_i: move the i-th book to the right

A typical self-optimizing model for:

- Cache handling
- Prioritizing

A first example: the Tsetlin library

Configuration: n books on a shelf
Operation T_i: move the i-th book to the right

A typical self-optimizing model for:

- Cache handling
- Prioritizing

Problem

- *Average behavior?*
- *How fast does it stabilize?*

Controlling the behavior of the Tsetlin library?

Markov chain description

- Configuration space Ω: all permutations of the books
- Transition operator T_i: taking book i with probability x_i

Controlling the behavior of the Tsetlin library?

Markov chain description

- Configuration space Ω: all permutations of the books
- Transition operator T_i: taking book i with probability x_i
- Transition operator $T = x_1 T_1 + \cdots + x_n T_n$

Controlling the behavior of the Tsetlin library?

Markov chain description

- Configuration space Ω: all permutations of the books
- Transition operator T_i: taking book i with probability x_i
- Transition operator $T = x_1 T_1 + \cdots + x_n T_n$
- Stationary distribution?
  1-Eigenvector of T

Controlling the behavior of the Tsetlin library?

Markov chain description

- Configuration space Ω: all permutations of the books
- Transition operator T_i: taking book i with probability x_i
- Transition operator $T = x_1 T_1 + \cdots + x_n T_n$
- Stationary distribution?
  1-Eigenvector of T
- Spectrum?
  Eigenvalues of T

Controlling the behavior of the Tsetlin library?

Markov chain description

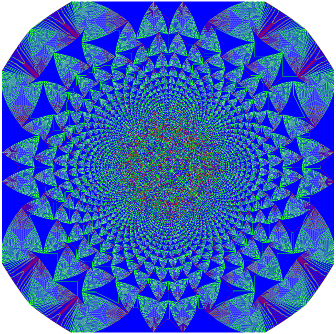
- Configuration space Ω: all permutations of the books
- Transition operator T_i: taking book i with probability x_i
- Transition operator $T = x_1 T_1 + \cdots + x_n T_n$
- Stationary distribution?
  1-Eigenvector of T
- Spectrum?
  Eigenvalues of T

Theorem (Brown, Bidigare '99)

Each $S \subseteq \{1, \ldots, n\}$ contributes the eigenvalue $\sum_{i \in S} x_i$ with multiplicity the number of **derangements** *of S.*

Abelian sandpile models / chip-firing games

- A graph G
- Configuration: distribution of grains of sand at each site
- Grains fall in at random
- Grains topple to the neighbor sites
- Grains fall off at sinks



- Prototypical model for the phenomenon of self-organized criticality, like a heap of sand

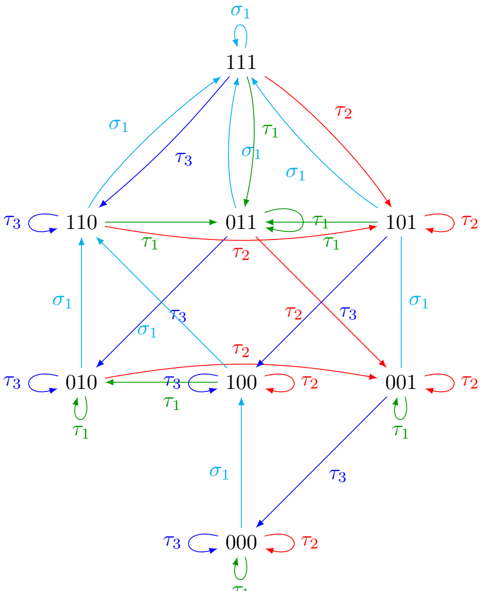
Directed sandpile Models

- A tree, with edges pointing toward its root
- Configuration: distribution of grains of sand at each site
- Grains fall in at random (leaves only or everywhere)
- Grains topple down at random (one by one or all at once)
- Grains fall off at the sink (=root)

Directed sandpile Models

- A tree, with edges pointing toward its root
- Configuration: distribution of grains of sand at each site
- Grains fall in at random (leaves only or everywhere)
- Grains topple down at random (one by one or all at once)
- Grains fall off at the sink (=root)
- System with reservoirs in nonequilibrium statistical physics

Directed sandpile model on a line with thresholds 1



Directed sandpile models are very nicely behaved

Proposition (Ayyer, Schilling, Steinberg, T. '13)

The transition graph is strongly connected
Equivalently the Markov chain is ergodic

Directed sandpile models are very nicely behaved

Proposition (Ayyer, Schilling, Steinberg, T. '13)

The transition graph is strongly connected
Equivalently the Markov chain is ergodic

Theorem (ASST'13)

Characteristic polynomial of the transition matrix:

$$\det(M_\tau - \lambda 1) = \prod_{S \subseteq V} (\lambda - (y_S + x_S))^{T_{S^c}}$$

where $S^c = V \setminus S$ *and* $T_S = \prod_{v \in S} T_v$

Directed sandpile models are very nicely behaved

Proposition (Ayyer, Schilling, Steinberg, T. '13)

The transition graph is strongly connected
Equivalently the Markov chain is ergodic

Theorem (ASST'13)

Characteristic polynomial of the transition matrix:

$$\det(M_\tau - \lambda 1) = \prod_{S \subseteq V} (\lambda - (y_S + x_S))^{T_{S^c}}$$

where $S^c = V \setminus S$ *and* $T_S = \prod_{v \in S} T_v$

Theorem (ASST'13)

Mixing time: at most $\frac{2(n_T + c - 1)}{p}$

Punchline

Those models have exceptionally nice eigenvalues

Punchline

Those models have exceptionally nice eigenvalues

In fact quite a few Markov chains have similar behaviors:

- Promotion Markov chains [Ayyer, Klee, Schilling '12]
- Nonabelian directed sandpile models
- Generalizations of the Tsetlin library (multibook, ...)
- Walks on longest words of finite Coxeter groups
- Half-regular bands

Punchline

Those models have exceptionally nice eigenvalues

In fact quite a few Markov chains have similar behaviors:

- Promotion Markov chains [Ayyer, Klee, Schilling '12]
- Nonabelian directed sandpile models
- Generalizations of the Tsetlin library (multibook, ...)
- Walks on longest words of finite Coxeter groups
- Half-regular bands

Is there some uniform explanation?

Punchline

Those models have exceptionally nice eigenvalues

In fact quite a few Markov chains have similar behaviors:

- Promotion Markov chains [Ayyer, Klee, Schilling '12]
- Nonabelian directed sandpile models
- Generalizations of the Tsetlin library (multibook, ...)
- Walks on longest words of finite Coxeter groups
- Half-regular bands

Is there some uniform explanation?

Yes

Punchline

Those models have exceptionally nice eigenvalues

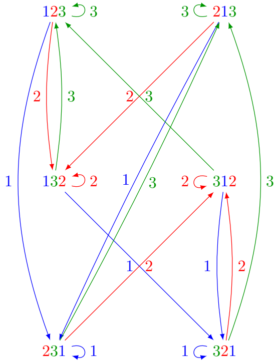
In fact quite a few Markov chains have similar behaviors:

- Promotion Markov chains [Ayyer, Klee, Schilling '12]
- Nonabelian directed sandpile models
- Generalizations of the Tsetlin library (multibook, ...)
- Walks on longest words of finite Coxeter groups
- Half-regular bands

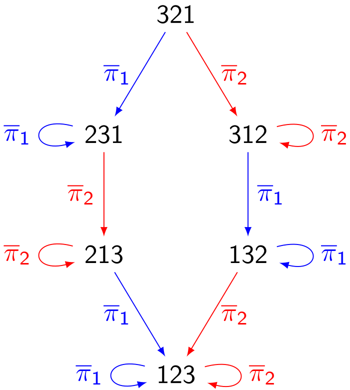
Is there some uniform explanation?

Yes: representation theory of $\mathcal{R}$-trivial monoids!

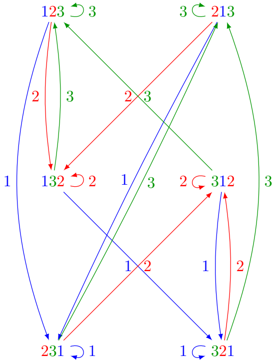
Decomposition of the configuration space (lumping)



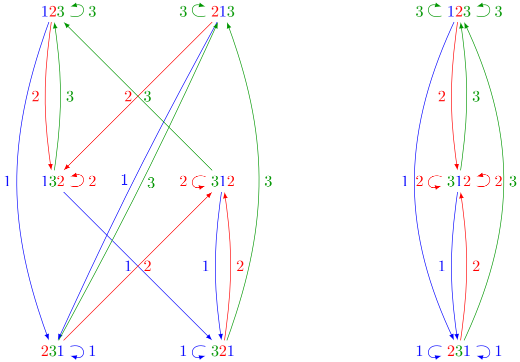
Let's train on a simpler example



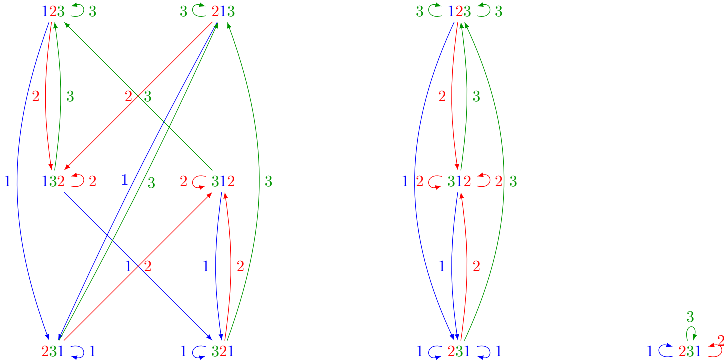
Decomposition of the configuration space (lumping)



Decomposition of the configuration space (lumping)



Decomposition of the configuration space (lumping)



Explanation: triangularization

Our Markov chains are nicely behaved because

- The transition operators T_i can be simultaneously triangularized!

Explanation: triangularization

Our Markov chains are nicely behaved because

- The transition operators T_i can be simultaneously triangularized!
- Divide, and conquer subspaces of dimension 1!

Explanation: triangularization

Our Markov chains are nicely behaved because

- The transition operators T_i can be simultaneously triangularized!
- Divide, and conquer subspaces of dimension 1!

Problems

- *Proving that it is triangularizable?*
- *Constructing a triangularization?*

Explanation: triangularization

Our Markov chains are nicely behaved because

- The transition operators T_i can be simultaneously triangularized!
- Divide, and conquer subspaces of dimension 1!

Problems

- *Proving that it is triangularizable?*
- *Constructing a triangularization?*

Or just be lazy

Learn a bit of representation theory and use characters.

Approach 2: monoids, representation theory, characters

Definition (Transition monoid of a Markov chain / automaton)

$T_i : \Omega \mapsto \Omega$ *transition operators* of the Markov chain

Approach 2: monoids, representation theory, characters

Definition (Transition monoid of a Markov chain / automaton)

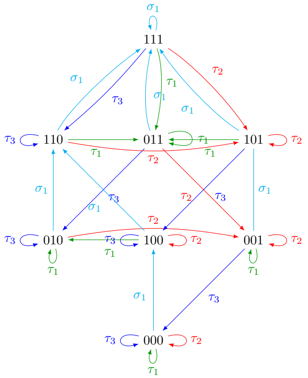
$T_i : \Omega \mapsto \Omega$ *transition operators* of the Markov chain

Monoid: $(\mathcal{M}, \circ) = \langle T_i \rangle$

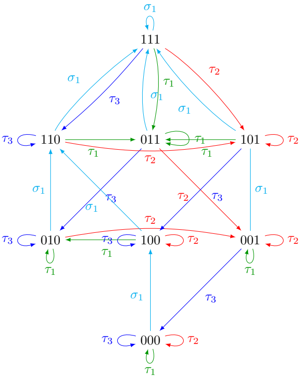
A finite monoid of functions

Similar to a permutation group, except for ~~invertibility~~

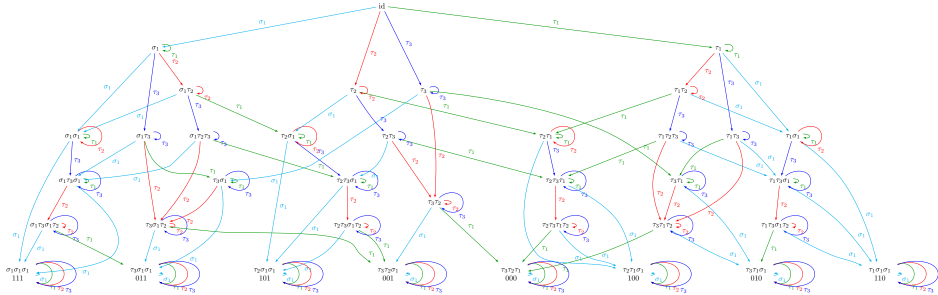
The left Cayley graph for the 1D sandpile model



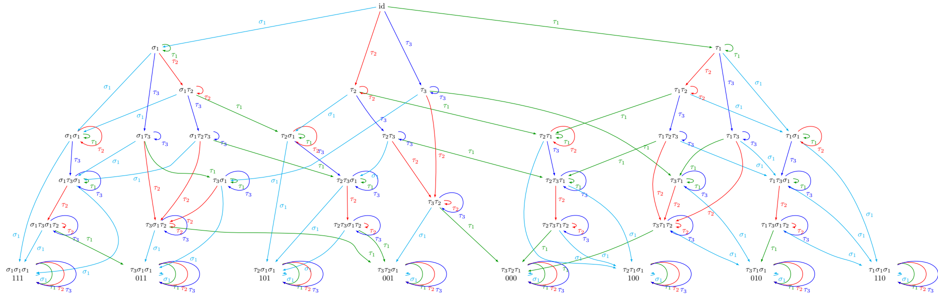
The left Cayley graph for the 1D sandpile model



The right Cayley graph for the 1D sandpile model

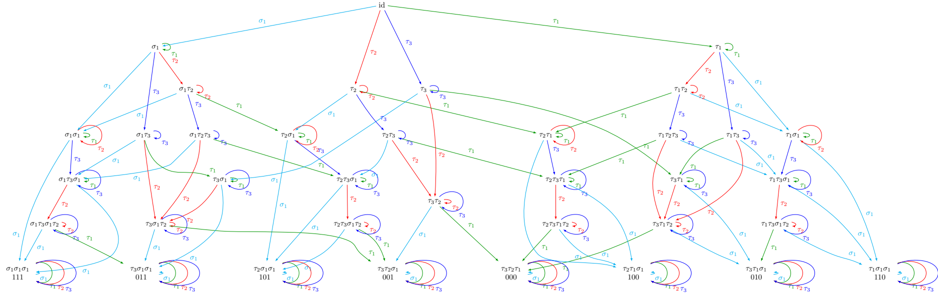


The right Cayley graph for the 1D sandpile model



- This graph is acyclic: $\mathcal{R}$-triviality

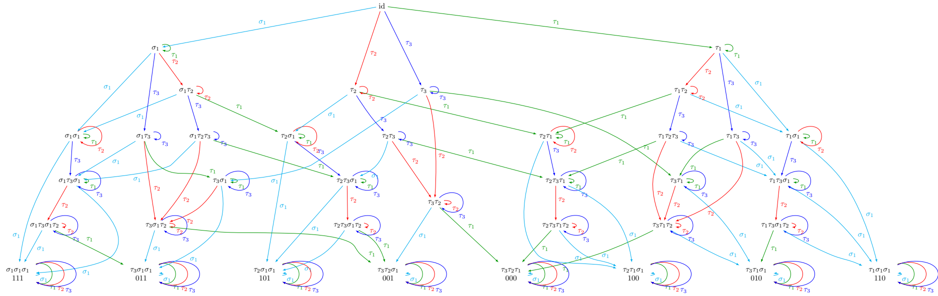
The right Cayley graph for the 1D sandpile model



- This graph is acyclic: $\mathcal{R}$-triviality

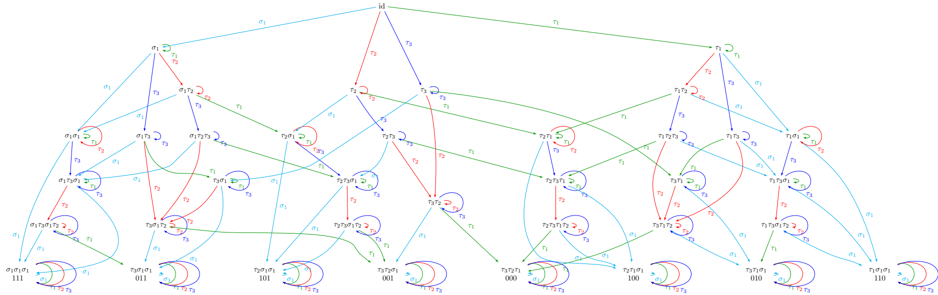
  $\Longrightarrow$ Triangularizability $\Longrightarrow$ Nice eigenvalues

The right Cayley graph for the 1D sandpile model



- This graph is acyclic: $\mathcal{R}$-triviality
  $\Longrightarrow$ Triangularizability $\Longrightarrow$ Nice eigenvalues
- What this captures: loss of information

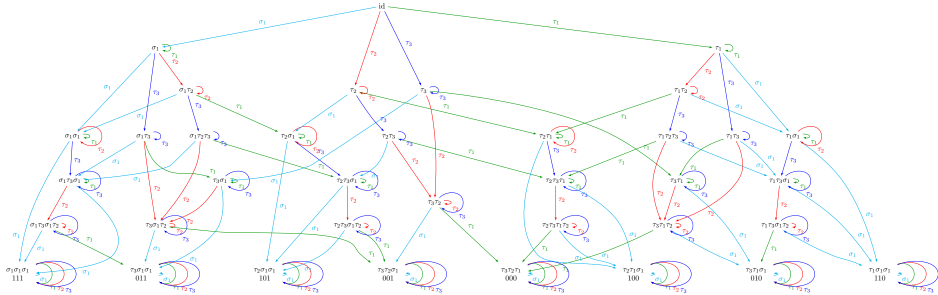
The right Cayley graph for the 1D sandpile model



- This graph is acyclic: $\mathcal{R}$-triviality

  $\Longrightarrow$ Triangularizability $\Longrightarrow$ Nice eigenvalues
- What this captures: loss of information
- Not too deep

The right Cayley graph for the 1D sandpile model

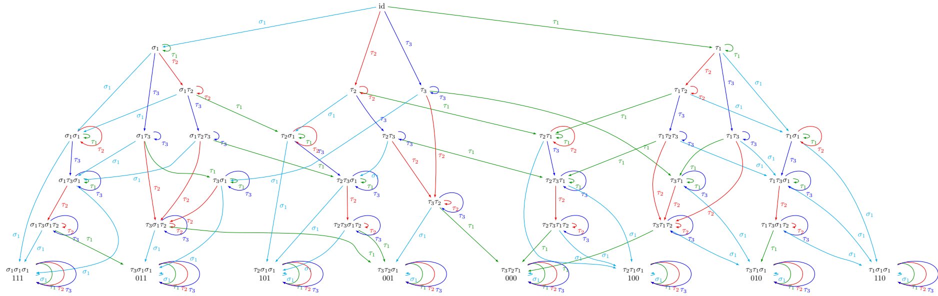


- This graph is acyclic: $\mathcal{R}$-triviality

  $\implies$ Triangularizability $\implies$ Nice eigenvalues
- What this captures: loss of information
- Not too deep

  $\implies$ Bound on the rates of convergence / mixing time

The right Cayley graph for the 1D sandpile model



- This graph is acyclic: $\mathcal{R}$-triviality
  $\Longrightarrow$ Triangularizability $\Longrightarrow$ Nice eigenvalues
- What this captures: loss of information
- Not too deep
  $\Longrightarrow$ Bound on the rates of convergence / mixing time
- A form of *coupling from the past*

Strategy

Combinatorial point of view

- Show that $\mathcal{M}$ is *$\mathcal{R}$-trivial*
  $\Rightarrow$ the representation matrix are simultaneously triangularizable

Strategy

Combinatorial point of view

- Show that $\mathcal{M}$ is *$\mathcal{R}$-trivial*
  $\Rightarrow$ the representation matrix are simultaneously triangularizable
- Eigenvalues indexed by a lattice of subsets of the generators

Strategy

Combinatorial point of view

- Show that $\mathcal{M}$ is *$\mathcal{R}$-trivial*
  $\Rightarrow$ the representation matrix are simultaneously triangularizable
- Eigenvalues indexed by a lattice of subsets of the generators
- Count fixed points
- Recover multiplicities from Möbius inversion on the lattice

Strategy

Combinatorial point of view

- Show that $\mathcal{M}$ is *$\mathcal{R}$-trivial*
  $\Rightarrow$ the representation matrix are simultaneously triangularizable
- Eigenvalues indexed by a lattice of subsets of the generators
- Count fixed points
- Recover multiplicities from Möbius inversion on the lattice

Representation theory point of view

- Simple modules are of dimension 1

Strategy

Combinatorial point of view

- Show that $\mathcal{M}$ is *$\mathcal{R}$-trivial*
  $\Rightarrow$ the representation matrix are simultaneously triangularizable
- Eigenvalues indexed by a lattice of subsets of the generators
- Count fixed points
- Recover multiplicities from Möbius inversion on the lattice

Representation theory point of view

- Simple modules are of dimension 1
- Compute the character of a transformation module

Strategy

Combinatorial point of view

- Show that $\mathcal{M}$ is *$\mathcal{R}$-trivial*
  $\Rightarrow$ the representation matrix are simultaneously triangularizable
- Eigenvalues indexed by a lattice of subsets of the generators
- Count fixed points
- Recover multiplicities from Möbius inversion on the lattice

Representation theory point of view

- Simple modules are of dimension 1
- Compute the character of a transformation module
- Recover the composition factors using the character table

This is effective!

GAP, Semigroupe

- Transformation monoids

Sage

- Character calculation for R-trivial (aperiodic) monoids
- Eigenvalues calculation
- Status: functional but not yet integrated
- Feel free to ask for a demo

This is effective!

GAP, Semigroupe

- Transformation monoids

Sage

- Character calculation for R-trivial (aperiodic) monoids
- Eigenvalues calculation
- Status: functional but not yet integrated
- Feel free to ask for a demo

Speaking of which

Interested in sharing code for studying Finite State Markov chains?
Let's talk! (today 2pm?)

How to prove $\mathbb{R}$-triviality?

How to prove $\mathbb{R}$-triviality?

Theorem (ASST'14)

Let M be a monoid generated by $A := \{x_1 < \cdots < x_n\}$ such that

$$a^2 = a \quad \text{for } a \in A \qquad\qquad bab = ba \quad \text{for } a < b \in A$$

Then, M is $\mathbb{R}$-trivial

How to prove $\mathbb{R}$-triviality?

Theorem (ASST'14)

Let M be a monoid generated by $A := \{x_1 < \cdots < x_n\}$ such that

$$a^2 = a \quad \text{for } a \in A \qquad\qquad bab = ba \quad \text{for } a < b \in A$$

Then, M is $\mathbb{R}$-trivial

Examples

Basically all our monoids!

How to prove $\mathbb{R}$-triviality?

Theorem (ASST'14)

Let M be a monoid generated by $A := \{x_1 < \cdots < x_n\}$ such that

$$a^2 = a \quad \text{for } a \in A \qquad\qquad bab = ba \quad \text{for } a < b \in A$$

Then, M is $\mathbb{R}$-trivial

Examples

Basically all our monoids!

Remark (Connection with the plactic monoid)

Take the (half) plactic monoid: $bac = bca$, for $a < b \leq c$
Set the generators to be idempotent: $a^2 = a$
Byproduct: $bab = bba = ba$

How to prove $\mathbb{R}$-triviality?

Theorem (ASST'14)

Let M be a monoid generated by $A := \{x_1 < \cdots < x_n\}$ *such that*

$$a^2 = a \quad \text{for } a \in A \qquad\qquad bab = ba \quad \text{for } a < b \in A$$

Then, M is $\mathbb{R}$*-trivial*

Examples

Basically all our monoids!

Remark (Connection with the plactic monoid)

Take the (half) plactic monoid: $bac = bca$, *for* $a < b \leq c$
Set the generators to be idempotent: $a^2 = a$
Byproduct: $bab = bba = ba$

Proof of the theorem.

Consider the worst case!

Free tree monoids

Definition (Free tree monoid M_n)

The monoid M_n generated by $A := \{x_1 < \cdots < x_n\}$ satisfying

$$a^2 = a \quad \text{for } a \in A \qquad\qquad bab = ba \quad \text{for } a < b \in A$$

Free tree monoids

Definition (Free tree monoid M_n)

The monoid M_n generated by $A := \{x_1 < \cdots < x_n\}$ satisfying

$$a^2 = a \quad \text{for } a \in A \qquad\qquad bab = ba \quad \text{for } a < b \in A$$

Proposition (ASST'14)

M_n is a monoid on a certain class of trees (#A007018)

Free tree monoids

Definition (Free tree monoid M_n)

The monoid M_n generated by $A := \{x_1 < \cdots < x_n\}$ satisfying

$$a^2 = a \quad \text{for } a \in A \qquad\qquad bab = ba \quad \text{for } a < b \in A$$

Proposition (ASST'14)

M_n is a monoid on a certain class of trees (#A007018)

Proof: reduced words + bijection with trees.

$x_3 x_2 x_4 x_1 x_2 x_4$

Free tree monoids

Definition (Free tree monoid M_n)

The monoid M_n generated by $A := \{x_1 < \cdots < x_n\}$ satisfying

$$a^2 = a \quad \text{for } a \in A \qquad\qquad bab = ba \quad \text{for } a < b \in A$$

Proposition (ASST'14)

M_n is a monoid on a certain class of trees (#A007018)

Proof: reduced words + bijection with trees.

$$x_3x_2x_4x_1x_2x_4 = x_3x_2x_4x_1x_2$$

Free tree monoids

Definition (Free tree monoid M_n)

The monoid M_n generated by $A := \{x_1 < \cdots < x_n\}$ satisfying

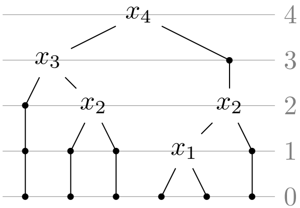
$$a^2 = a \quad \text{for } a \in A \qquad\qquad bab = ba \quad \text{for } a < b \in A$$

Proposition (ASST'14)

M_n is a monoid on a certain class of trees (#A007018)

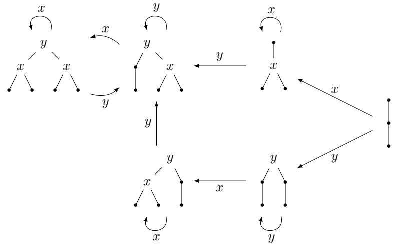
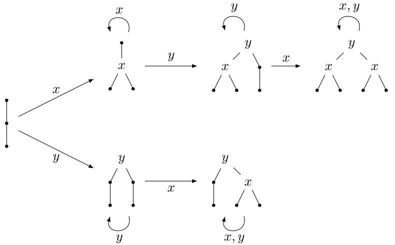
Proof: reduced words + bijection with trees.

$$x_3x_2x_4x_1x_2x_4 = x_3x_2x_4x_1x_2 \quad \longleftrightarrow$$



□

Free tree monoids

Left Cayley graph	Right Cayley graph
	

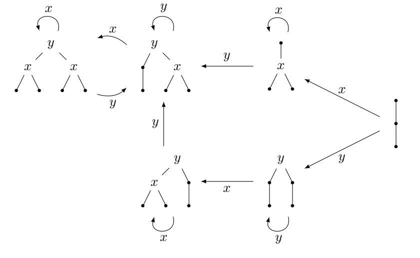
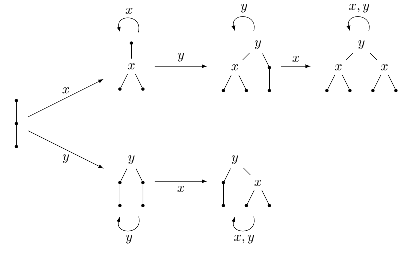
Proposition (ASST'14)

M_n is $\mathbb{R}$-trivial

Proof.

Nice explicit Knuth-Bendix completion of the relations □

Free tree monoids

Left Cayley graph	Right Cayley graph
	

Proposition (ASST'14)

M_n is $\mathbb{R}$-trivial

Proof.

Nice explicit Knuth-Bendix completion of the relations □

Problem

Description of the stationary distribution of the left Cayley graph?

Finite state Markov chains and Representation Theory

The idea of decomposing the configuration space is not new!

Using representation theory of groups

- Diaconis et al.
- Nice combinatorics (symmetric functions, ...)

Finite state Markov chains and Representation Theory

The idea of decomposing the configuration space is not new!

Using representation theory of groups

- Diaconis et al.
- Nice combinatorics (symmetric functions, ...)

Using representation theory of monoids

- Any finite state Markov chain can be seen as a representation of a monoid M
- M can be chosen to be a group iff the uniform distribution is stationary.

Finite State Markov chains and Representation Theory

Using representation theory of right regular bands

- Tsetlin library, Hyperplane arrangements, ...
- Bidigare, Hanlon, Rockmore '99, Brown '00, Saliola, ...
- Revived the interest for representation theory of monoids

Finite State Markov chains and Representation Theory

Using representation theory of right regular bands

- Tsetlin library, Hyperplane arrangements, ...
- Bidigare, Hanlon, Rockmore '99, Brown '00, Saliola, ...
- Revived the interest for representation theory of monoids

Using representation of $\mathcal{R}$-trivial monoids?

- Steinberg '06, ...
- Not semi-simple.

Finite State Markov chains and Representation Theory

Using representation theory of right regular bands

- Tsetlin library, Hyperplane arrangements, ...
- Bidigare, Hanlon, Rockmore '99, Brown '00, Saliola, ...
- Revived the interest for representation theory of monoids

Using representation of $\mathcal{R}$-trivial monoids?

- Steinberg '06, ...
- Not semi-simple. But simple modules of dimension 1!
- Nice combinatorics

Punchlines

- A (useful?) class of Markov chains with very nice behavior

Punchlines

- A (useful?) class of Markov chains with very nice behavior
  Explanation: representation theory of monoids
- Representation theory of monoids $\Longrightarrow$ Nice combinatorics
- Tool: computer exploration

Punchlines

- A (useful?) class of Markov chains with very nice behavior
  Explanation: representation theory of monoids
- Representation theory of monoids $\Longrightarrow$ Nice combinatorics
- Tool: computer exploration

Announcements

- Tentative European H2020 Sage proposal
  Want to participate?

Punchlines

- A (useful?) class of Markov chains with very nice behavior
  Explanation: representation theory of monoids
- Representation theory of monoids $\Longrightarrow$ Nice combinatorics
- Tool: computer exploration

Announcements

- Tentative European H2020 Sage proposal
  Want to participate?
- Maître de conférence position
  GALAC Team: Graphes Algorithmes et Combinatoire
  Laboratoire de Recherche en Informatique
  Université Paris Sud