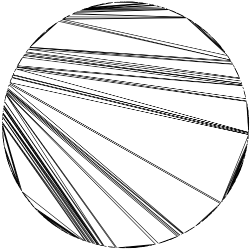
The dual tree of a recursive triangulation of the disk



Henning Sulzbach, INRIA Paris-Rocquencourt

Journées Alea, Luminy, March 2014

joint work with Nicolas Broutin (INRIA)

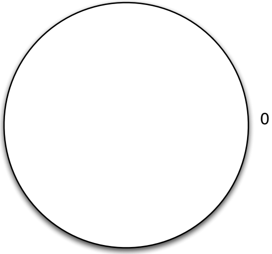
Outline

Outline

Recursive laminations of the disk

Curien and Le Gall 2011:

In each step, connect two uniformly chosen points unless the chord intersects any previously inserted.



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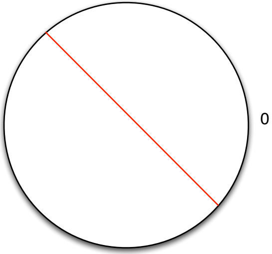

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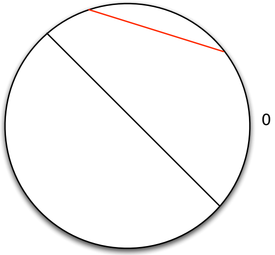

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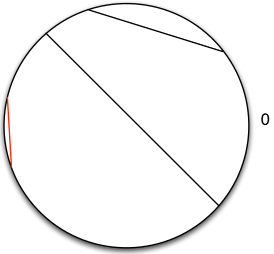

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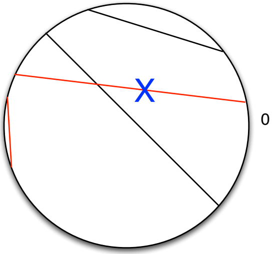

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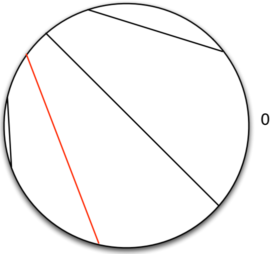

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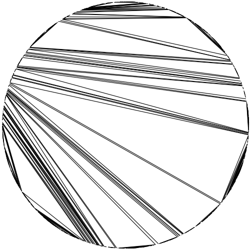
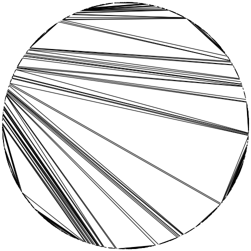


Number of inserted chords at time n is about $\sqrt{\pi n}$.

Lamination: $L_n =$ set of inserted chords at time n.

The limit triangulation

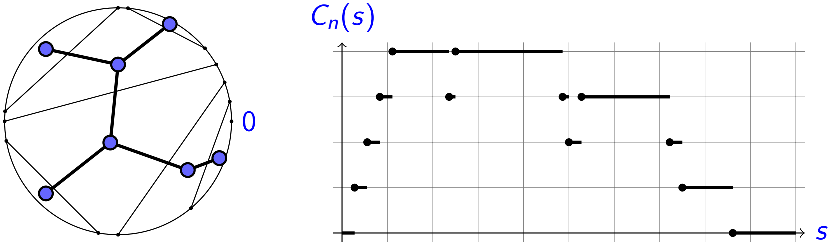


Theorem (Curien, Le Gall)

$\mathcal{L}_\infty := \overline{\bigcup_{n\geq 1} L_n}$ *is a triangulation, that is, its complement consists of triangles with vertices on the circle.*

Observe: Triangulations are maximal, that is, they cannot be increased by additional chords.

The dual tree



T_n: dual tree, d_{gr}: graph distance on T_n.

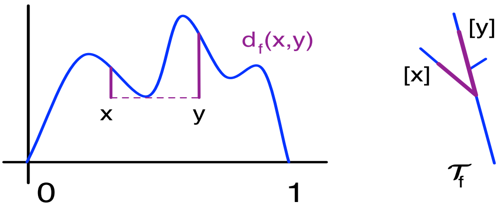
$C_n(s) =$ depth of node at $s \in [0, 1]$ in T_n.

Scaling limit of the dual tree T_n ?

Scaling limit of the contour process $C_n(s)$?

Trees encoded by excursions

Let $f : [0,1] \to \mathbb{R}^+$ be a continuous excursion.



$\mathcal{T}_f = {}^{[0,1]}/_{\sim}$ where $s \sim t$ with $s \leq t$ if $d_f(s,t) = 0$ where

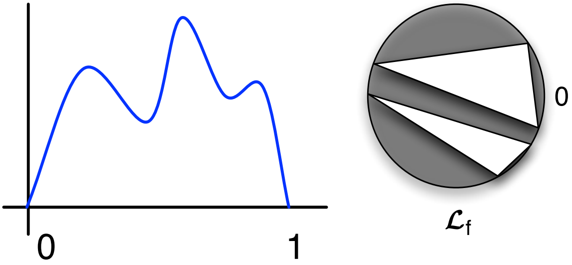
$$d_f(s,t) = f(s) + f(t) - 2\inf\{f(x) : s \leq x \leq t\}.$$

$(\mathcal{T}_f, d_f)$ is a compact tree-like metric space (an $\mathbb{R}$-tree).

Triangulations encoded by excursions

Let $f : [0,1] \to \mathbb{R}^+$ be a continuous excursion with *distinct* local minima.

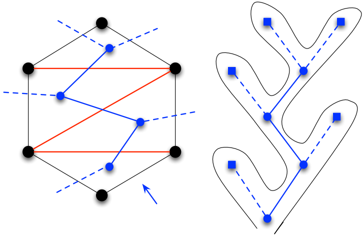
$\mathcal{L}_f$ contains chords connecting $s \le t$ if and only if $d_f(s,t) = 0$.

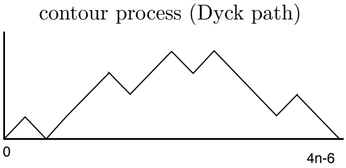


Inner nodes of $\mathcal{T}_f$ correspond to triangles in $\mathcal{L}_f$.

The Brownian world - Aldous '94

Consider uniform triangulations of the n-gon P_n:





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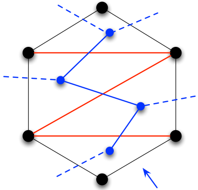
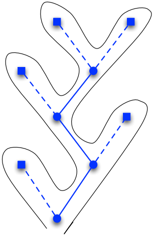
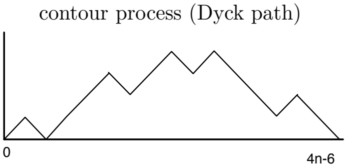
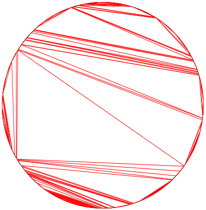
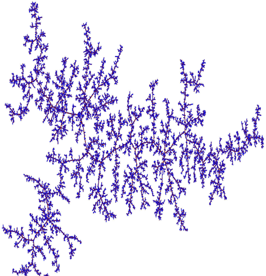


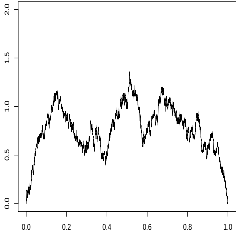




$\downarrow \delta_{\text{Haus}}$ $\downarrow d_{GH}$ $\downarrow \|\cdot\|_\infty$



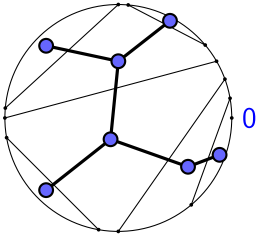


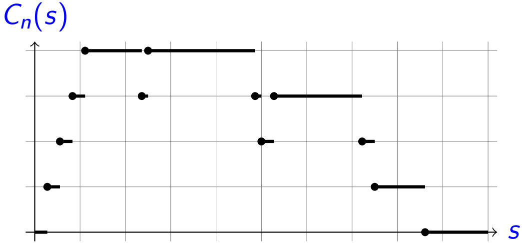


Outline

1. The model and its background
2. **Main results**
3. Variations of the scheme

The dual tree of the lamination





$C_n(s) =$ depth of node at $s \in [0,1]$ in T_n.

Theorem (Broutin, S. '14)

There exists a random continuous process $Z(s), s \in [0,1]$*, such that, uniformly in* $s \in [0,1]$*, almost surely,*

$$\frac{C_n(s)}{n^{\beta/2}} \to Z(s), \qquad \beta = \frac{\sqrt{17}-3}{2} = 0.561\ldots$$

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$$\frac{C_n(s)}{n^{\beta/2}} \to Z(s), \qquad \beta = \frac{\sqrt{17}-3}{2} = 0.561\ldots$$

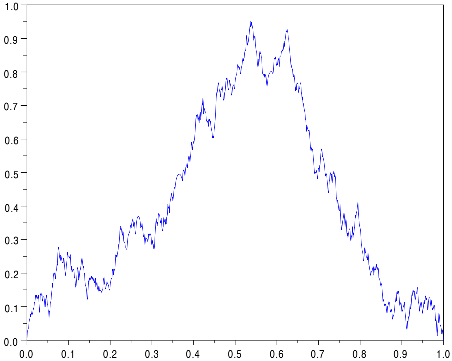
Moreover, $\mathcal{L}_\infty = \mathcal{L}_Z$ *(already proved by Curien and Le Gall).*

Almost surely,

$$(T_n, n^{-\beta/2} d_{gr}) \to (\mathcal{T}_Z, d_Z)$$

in the Gromov-Hausdorff topology on the space of (isometry classes of) compact metric spaces.

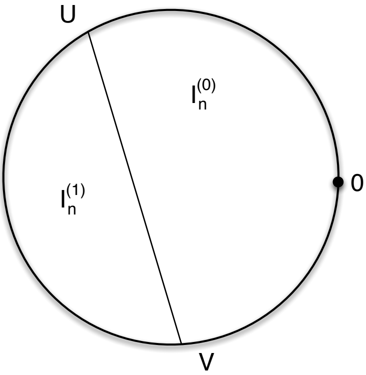
A simulation of the limit



$\mathbb{E}\left[Z(s)\right] \sim (s(1-s))^{\beta}$

Optimal Hölder exponent: $\beta = 0.561\ldots.$

Recursive decomposition

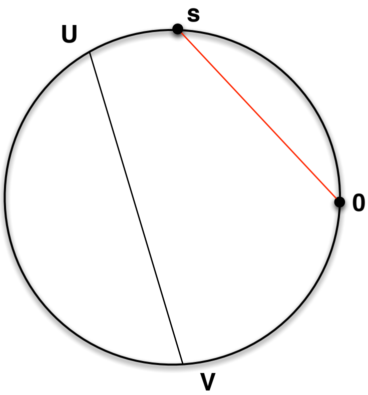


Attempted insertions in subfragments

$$I_n^{(0)} \stackrel{d}{=} \mathrm{Bin}(n-1,(1-(V-U))^2) \sim n(1-(V-U))^2$$

$$I_n^{(1)} \stackrel{d}{=} \mathrm{Bin}(n-1,(V-U)^2) \sim n(V-U)^2$$

Recursive decomposition



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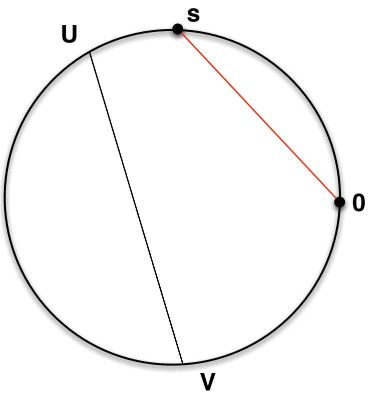
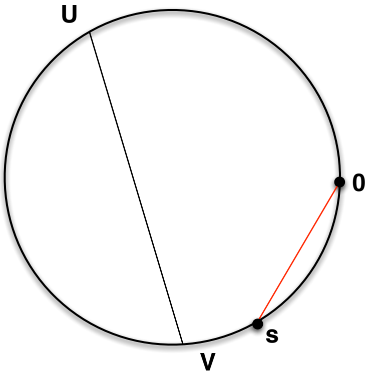


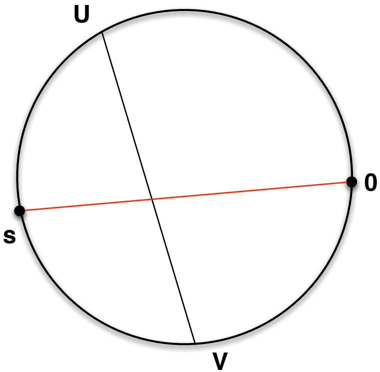
$$C_n(s) \stackrel{d}{=} \mathbf{1}_{[0,U]}(s)\, C^{(0)}_{I_0^{(n)}}\left(\frac{s}{1-(V-U)}\right)$$

Recursive decomposition



$$C_n(s) \stackrel{d}{=} \mathbf{1}_{[0,U]}(s) C^{(0)}_{I_0^{(n)}}\left(\frac{s}{1-(V-U)}\right) + \mathbf{1}_{(V,1]}(s) C^{(0)}_{I_0^{(n)}}\left(\frac{s-(V-U)}{1-(V-U)}\right)$$

Recursive decomposition



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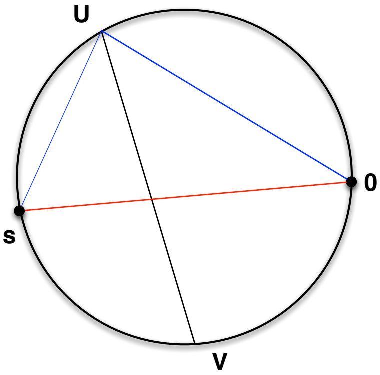

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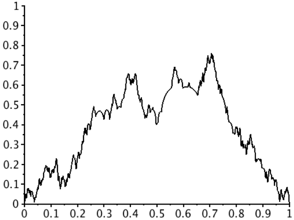
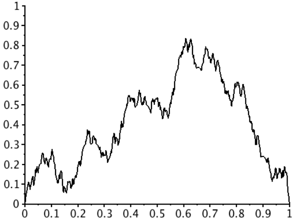
$$+ \mathbf{1}_{(U,V]}(s)\left(1 + C^{(0)}_{I_0^{(n)}}\left(\frac{U}{1-(V-U)}\right) + C^{(1)}_{I_1^{(n)}}\left(\frac{s-U}{V-U}\right)\right)$$

Characterizing Z

(U, V) min and max of two ind. uniforms, here $U = 0.32, V = 0.56$



$Z^{(0)}$



$Z^{(1)}$

Characterizing Z

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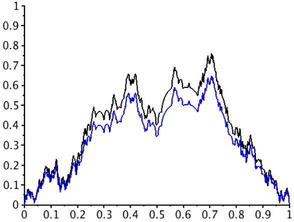
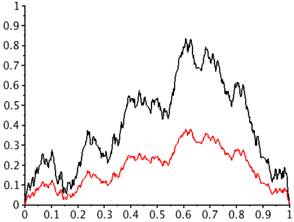


$Z^{(0)}$, $(1-(V-U))^{\beta} Z^{(0)}$



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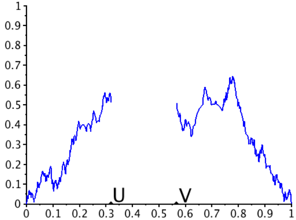

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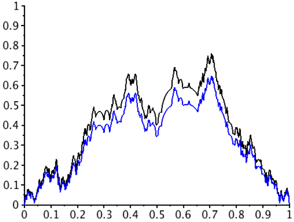

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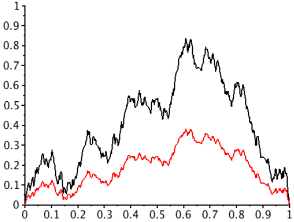

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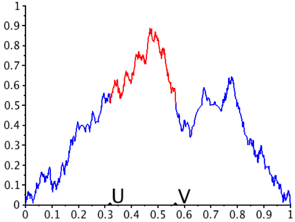


The fractal dimension

Theorem (Broutin, S.)

Almost surely, we have $\dim(\mathcal{T}_Z) = \frac{1}{\beta} = 1.781\ldots$ *both for Minkowski and Hausdorff dimension.*

Compare: $\dim(\mathcal{T}_e) = 2$ for the CRT.

Very roughly, $\dim(\mathcal{T}_f) = s$ means that, as $r \to 0$,

$$|B_r(x)| \approx r^s$$

with

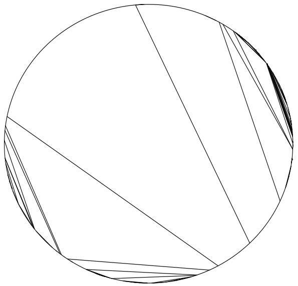
$$B_r(x) = \{y \in \mathcal{T}_f : d_f(x, y) < r\}.$$

Outline

A homogeneous model

In each step

- choose one fragment uniformly at random
- insert a chord uniformly at random



Observe: $I_0^{(n)}$ is uniformly distributed (Polya urn!), hence

$$\frac{I_0^{(n)}}{n} \to W, \quad n \to \infty,$$

where W is uniform on $[0, 1]$ and independent of (U, V).

A homogeneous model

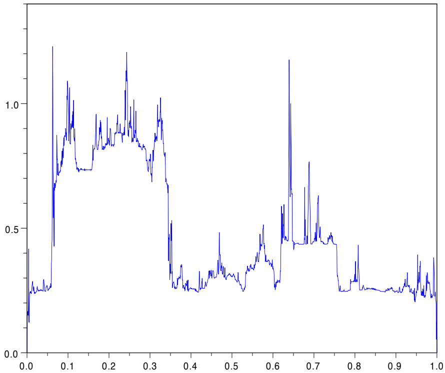
Theorem (Broutin, S. '14)

There exists a random continuous process $H(s), s \in [0,1]$*, such that, uniformly in* $s \in [0,1]$*, almost surely,*

$$\frac{C_n^h(s)}{n^{1/3}} \to H(s).$$

Moreover, $\mathbb{E}\left[H(s)\right] \sim (s(1-s))^{1/2}$.

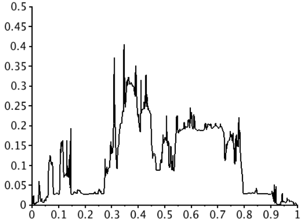
A simulation of H



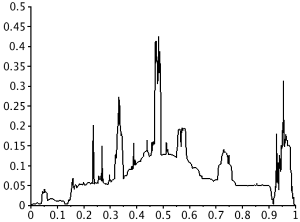
Optimal Hölder exponent: $\frac{3-2\sqrt{2}}{3} = 0.057\ldots$

The characterization of H

(U, V): as before and W another independent uniform.



$H^{(0)}$



$H^{(1)}$

The characterization of H

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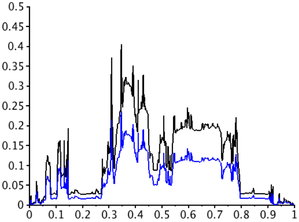
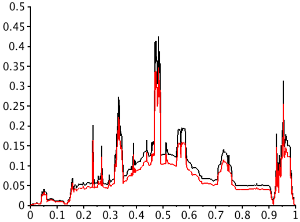


$H^{(0)}$, $(\mathbf{1}-\mathbf{W})^{1/3}H^{(0)}$



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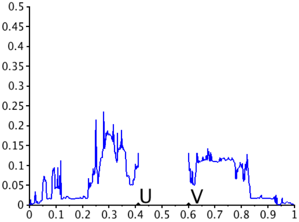

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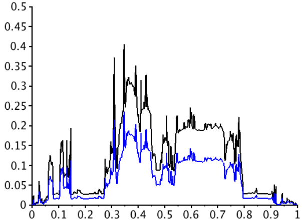

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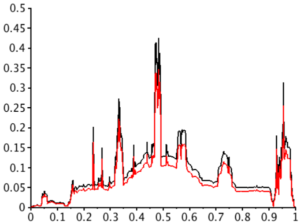

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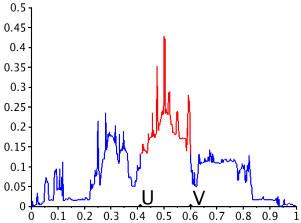


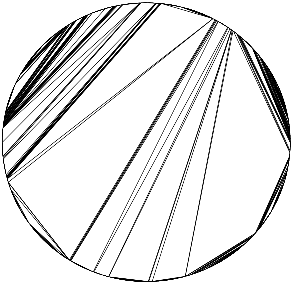
The fractal dimension

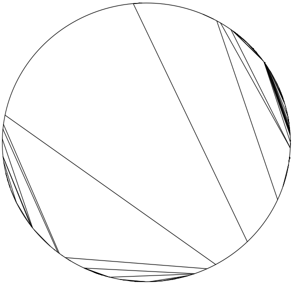
Theorem (Broutin, S.)

Almost surely, we have $\dim(\mathcal{T}_H) = 3$ *both for Minkowski and Hausdorff dimension.*

The last slide

Note: L_{n^2} and L_n^h and their dual trees are significantly different.





$$\frac{\log M_{n^2}}{\log n} \to -1 \qquad \frac{\log M_n^h}{\log n} \to \sqrt{8} - 3 = -0.18\ldots$$

On the other hand: $\mathcal{L}_H = \mathcal{L}_Z, \dim \mathcal{L}_Z = 1 + \beta$.

References

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- D. Aldous. Recursive self-similarity for random trees, random triangulations and Brownian excursion. *Ann. Probab.*, **22**: 527–545, 1994
- N. Curien and J.-F. Le Gall. Random recursive triangulations of the disk via fragmentation theory. *Ann. Probab.*, **39**: 2224-2270, 2011
- N. Broutin and H. Sulzbach. The dual tree of a recursive triangulation of the disk. *to appear in Ann. Probab.*, 2014

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