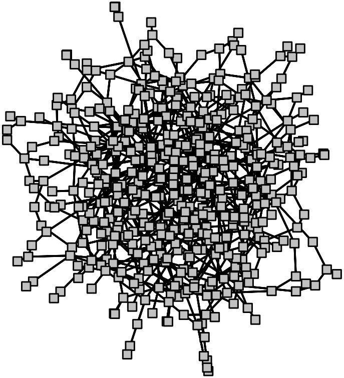
The densest subgraph of sparse random graphs



Justin Salez (Université Paris 7)
with Venkat Anantharam (UC Berkeley)

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- **Idea:** formalize that via *local weak convergence*, and use this framework to replace the asymptotic study of large graphs by the direct analysis of their infinite-volume limits.

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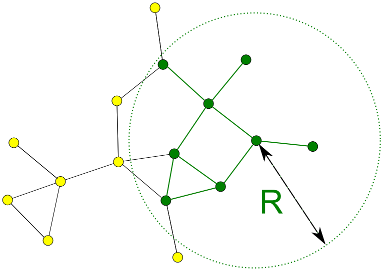

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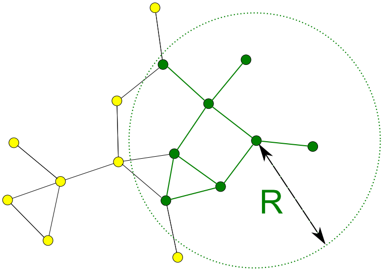

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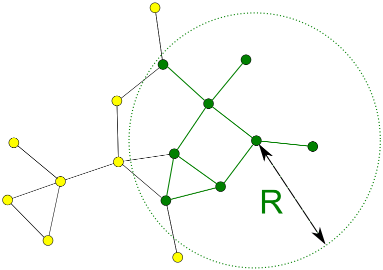

$(G_n, o_n) \xrightarrow[n\to\infty]{} (G, o)$ if for each **fixed** R, there is $n_R \in \mathbb{N}$ such that

$$n \geq n_R \implies [G_n, o_n]_R \equiv [G, o]_R$$

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- $\mathcal{L}$ describes the local geometry of G_n around a random node

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where $f(z) = \sum_n \pi_n z^n$ and $\lambda^* = f'(1-\lambda)/f'(1)$.

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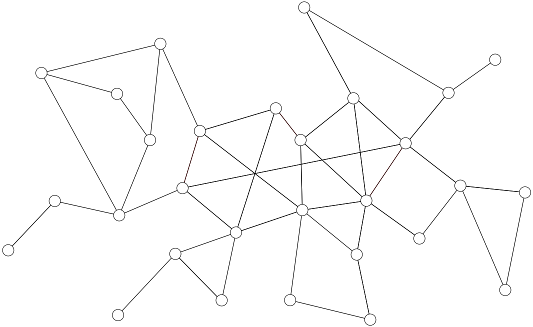

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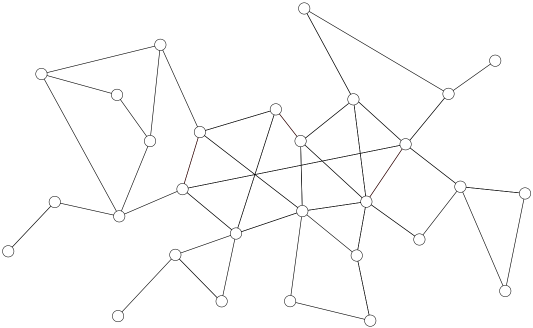

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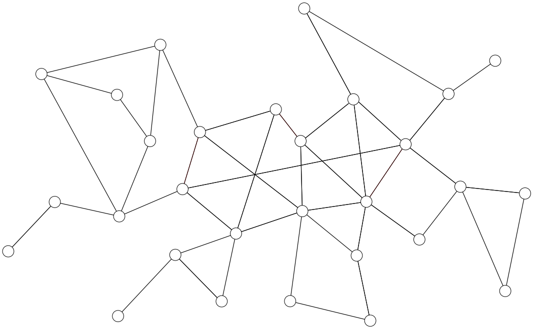

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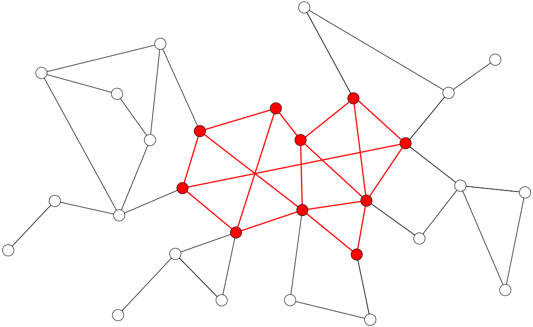

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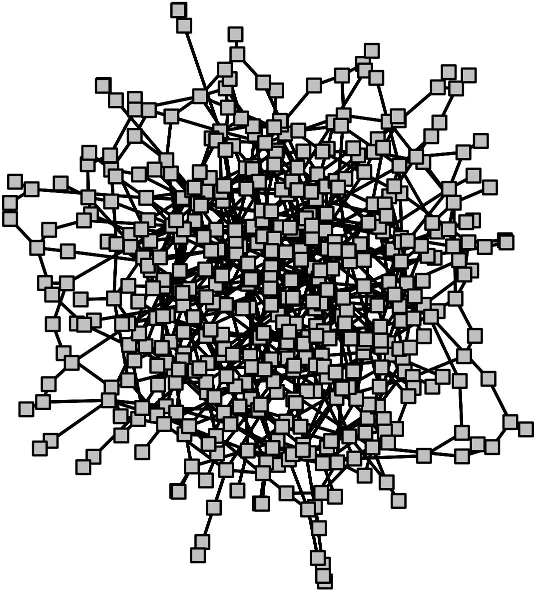

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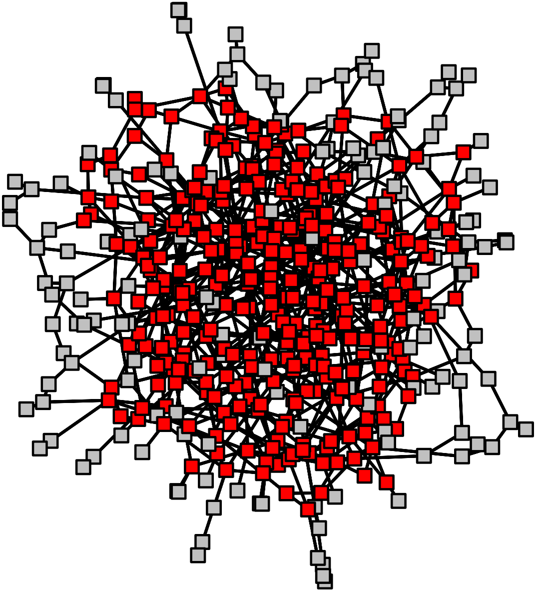

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An **allocation** on G is a function $\theta \colon \vec{E} \to [0,1]$ satisfying

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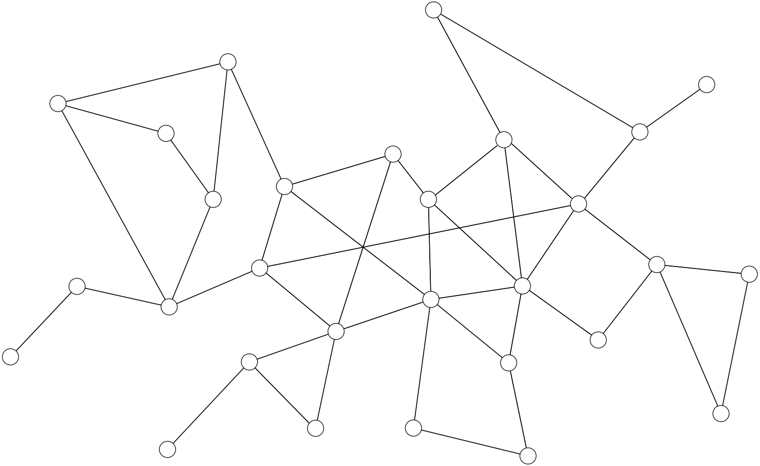
Example



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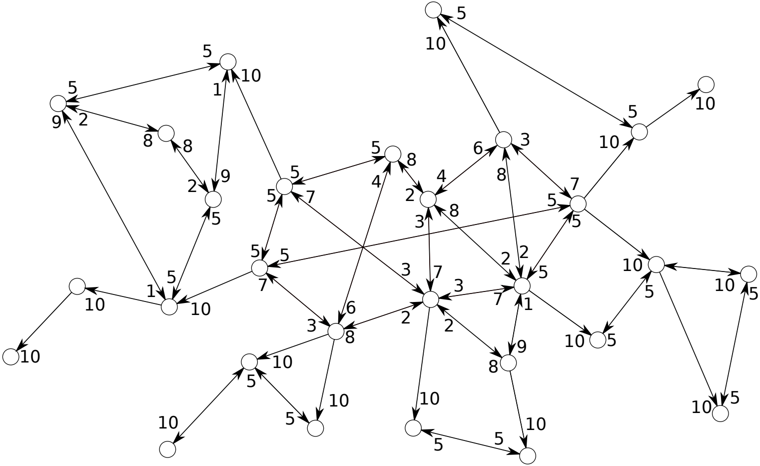

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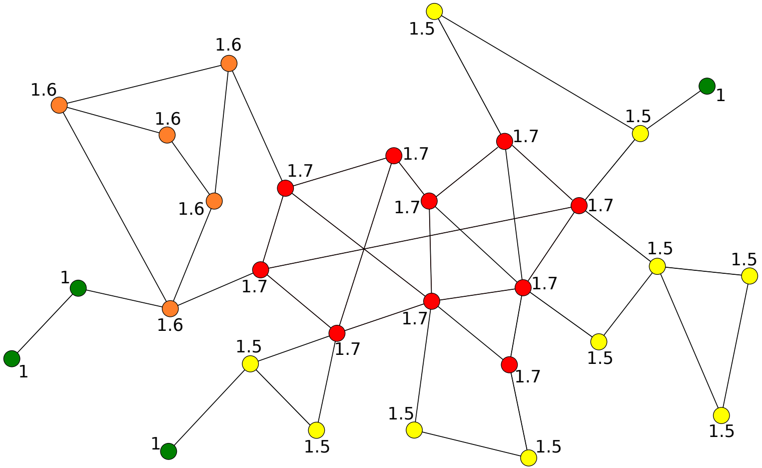
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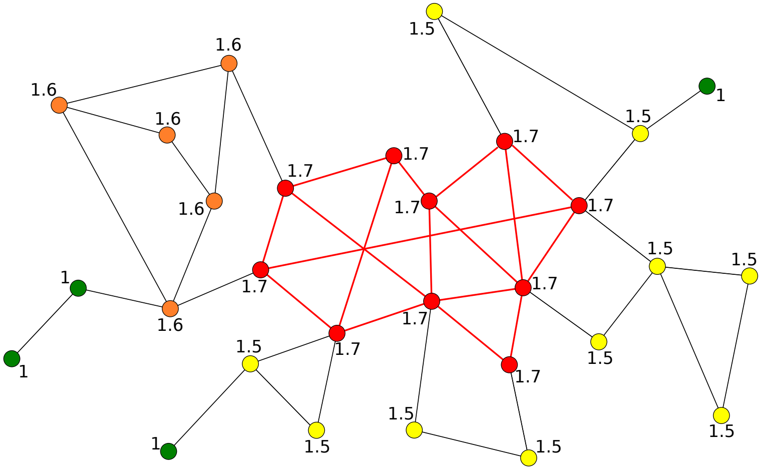
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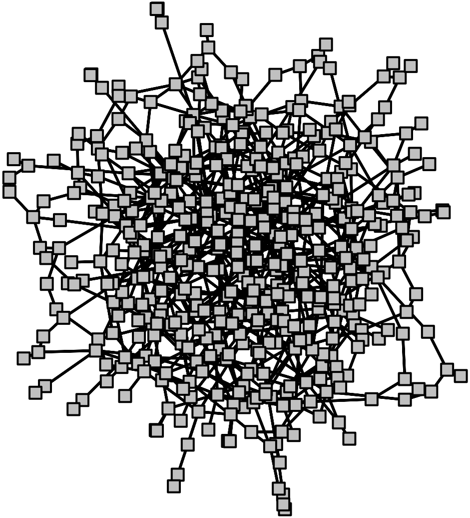

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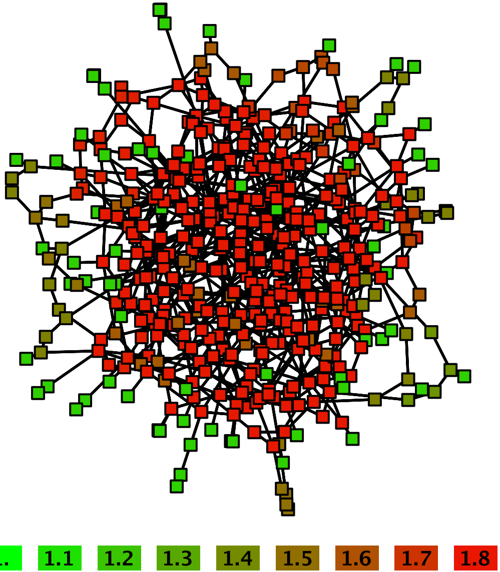

Density profile of a random graph with average degree 3

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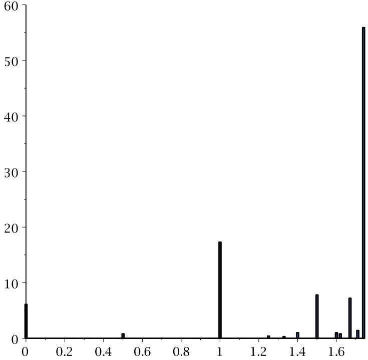

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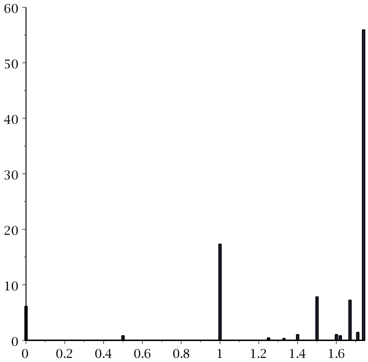

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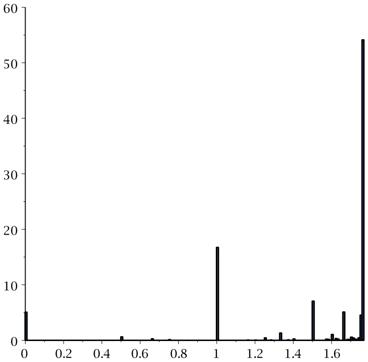

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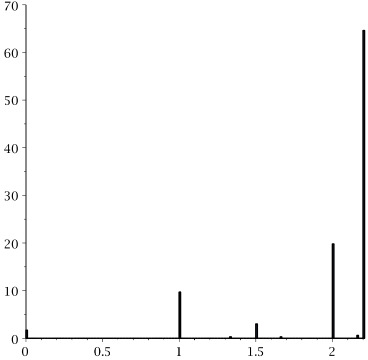

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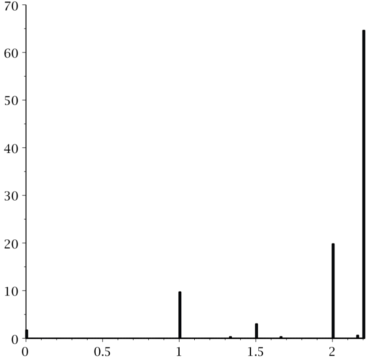

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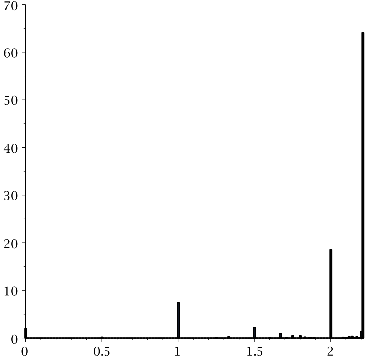

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Distributional fixed-point equation : can be solved numerically.

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k	2	3	4	5	6	7	8	9	10
c_*	3.59	5.76	7.84	9.90	11.93	13.95	15.97	17.98	19.98

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$$[G,o]_R \equiv [G',o']_R \implies \left|\partial\Theta(G,o) - \partial\Theta(G',o')\right| \le f(R),$$

where $f(R) \to 0$ as $R \to \infty$.

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▶ **Balanced loads exhibit long-range dependences !**

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Corollary. Θ_ε extends continuously to infinite graphs !

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- **Analytic implication:** Φ admits a limit along most sparse graph sequences. The distributional self-similarity of $\mathcal{L}$ may sometimes even allow for an explicit determination of $\Phi(\mathcal{L})$.
- **Many examples:** spanning trees, spectrum and rank, matching polynomial, Ising models, dense subgraphs...

Thank you !